

Stated Skein Modules of Marked 3–Manifolds

Wade Bloomquist (Georgia Tech)

GW Topology Seminar

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The One Slide Version

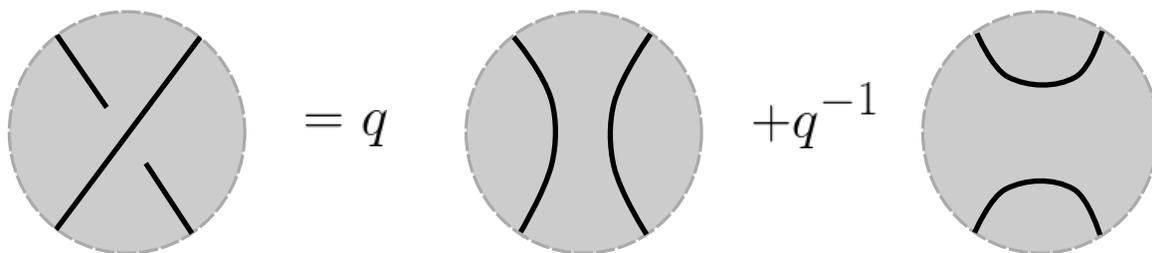
- We will work with a generalization of skein modules that adds extra data for the end points of tangles.
- This extra data allows for kinds of “building block structure” (and a rich underlying grounding in algebra)
- There are lessons to be learned by exploring skein module concepts in this generalized framework

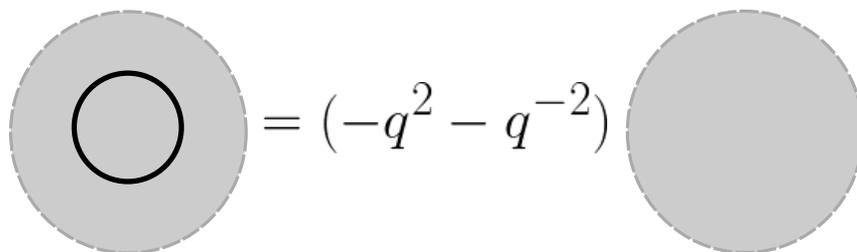
Outline

- 1 Skein Modules
- 2 Marked 3–Manifolds and Stated Skein Modules
- 3 The Chebyshev-Frobenius Homomorphism
- 4 Chebyshev-Frobenius Properties and Observations

The Kauffman Bracket Skein Module

Formal linear combinations of isotopy classes of framed links in a 3-manifold M (with coefficients in $\mathbb{Z}[q^\pm]$), subject to the following local moves:


$$\text{Crossing} = q \text{ (Separated)} + q^{-1} \text{ (Connected)}$$


$$\text{Circle with inner circle} = (-q^2 - q^{-2}) \text{ (Empty circle)}$$

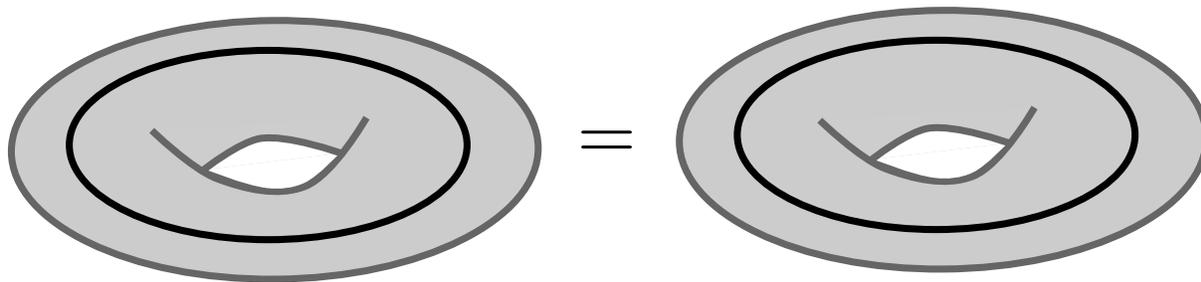
The Skein Module of S^3



$$= q^2(-q^2 - q^{-2})^2 + (-q^2 - q^{-2}) + (-q^2 - q^{-2}) + q^{-2}(-q^2 - q^{-2})^2$$

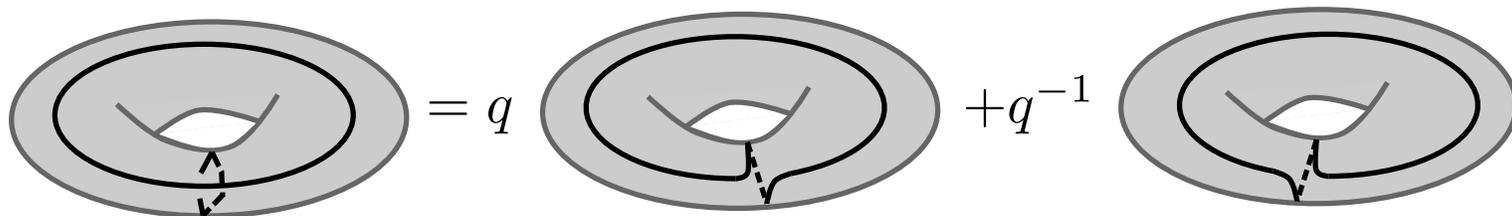
times the empty diagram

The Skein Module of The Genus One Handlebody



Skein Algebras of Surfaces

If Σ is an oriented surface then the skein module of $\Sigma \times I$ is an algebra with multiplication given by *vertical stacking* and resolving.



Theorem (Przytycki 1999)

The skein algebra of Σ is free with a basis given by simple diagrams of multicurves on the surface.

The Chebyshev Homomorphism

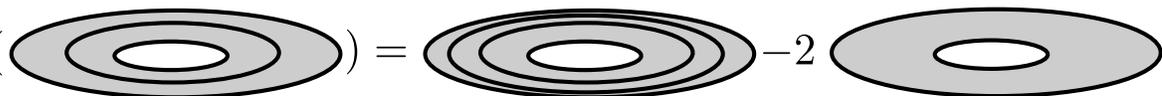
Recall the recursively defined

$$T_0(z) = 2, \quad T_1(z) = z, \quad T_n(z) = zT_{n-1}(z) - T_{n-2}(z)$$

Theorem (Bonahon and Wong 2012)

Let ω be a root of unity with $N = \text{ord}(\omega^8)$ and $\eta = \omega^{N^2}$. There is a map between the skein algebras of unmarked surfaces that sends any knot α to $T_N(\alpha)$.

Key in understanding the algebraic structure and representation theory of skein algebras.

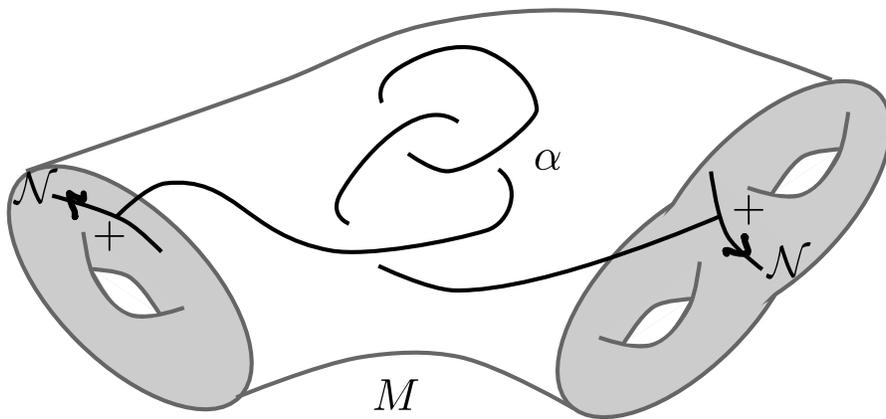
$$T_2(\text{torus}) = \text{torus} - 2 \text{torus}$$


Markings and States

A **marked 3-manifold**, (M, \mathcal{N}) , is a 3-manifold together with a set of “marking arcs” contained in ∂M .

A \mathcal{N} -**tangle** in M is a framed 1-manifold in M . Think framed links and framed arcs ending on pieces of \mathcal{N} .

A **state** for a \mathcal{N} -tangle, α , in M is an assignment $s : \alpha \cap \mathcal{N} \rightarrow \{\pm\}$.



The Stated Skein Module

Formal linear combinations of isotopy classes of **stated \mathcal{N} -tangles** in a marked 3-manifold (M, \mathcal{N}) (with coefficients in $\mathbb{Z}[q^\pm]$), subject to the Kauffman skein relations and the following boundary moves:

“Top View”

$$\begin{array}{c} \text{Loop on } -\mathcal{N} \text{ to } + \\ = q^{-1/2} \text{ Dot on } \mathcal{N} \end{array}$$

$$\begin{array}{c} \text{Loop on } +\mathcal{N} \text{ to } + \\ = 0 = \text{Loop on } -\mathcal{N} \text{ to } - \end{array}$$

$$\begin{array}{c} \text{Lines meeting at } -\mathcal{N} \text{ to } + \\ = q \text{ Lines meeting at } -\mathcal{N} \text{ to } + \end{array}$$

“Side View”

$$\begin{array}{c} \text{Loop on } \mathcal{N} \text{ to } + \\ = q^{-1/2} \text{ Dot on } \mathcal{N} \end{array}$$

$$\begin{array}{c} \text{Loop on } \mathcal{N} \text{ to } + \\ = 0 = \text{Loop on } \mathcal{N} \text{ to } - \end{array}$$

$$\begin{array}{c} \text{Lines meeting at } \mathcal{N} \text{ to } - \\ = q \text{ Lines meeting at } \mathcal{N} \text{ to } + \end{array}$$

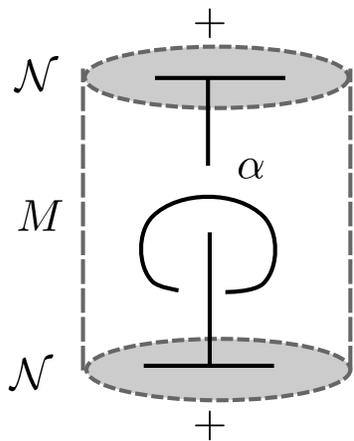
Functoriality (Think Locally)

If

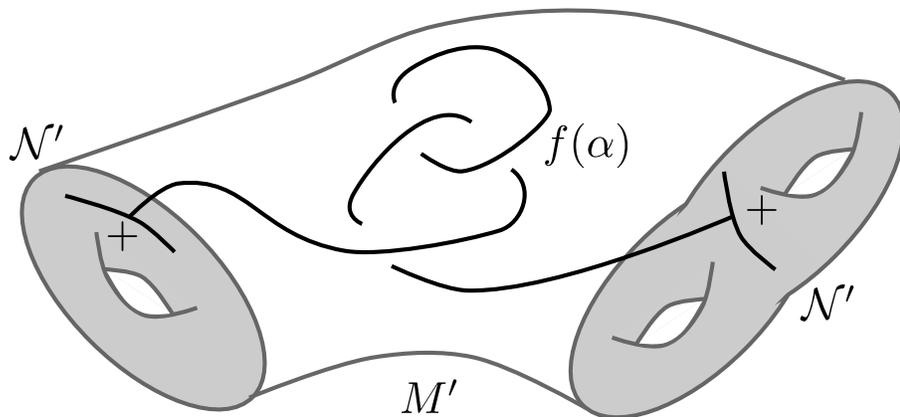
$$f : M \xrightarrow{\text{embedding}} M' \quad \text{with} \quad f|_{\mathcal{N}} : \mathcal{N} \xrightarrow{\text{embedding}} \mathcal{N}'$$

then we have a well defined module map

$$f_* : \mathcal{S}(M, \mathcal{N}) \rightarrow \mathcal{S}(M', \mathcal{N}') \quad \text{with} \quad f_*([\alpha]) = [f(\alpha)]$$

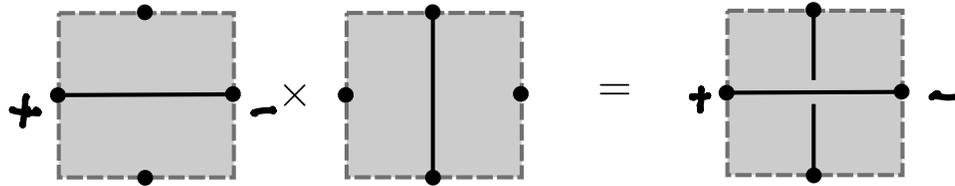


$f \rightarrow$



Marked Surfaces

If $M = \Sigma \times (-1, 1)$ and $\mathcal{N} = \mathcal{P} \times (-1, 1)$ for $\mathcal{P} \subseteq \partial\Sigma$, then *vertical stacking* defines a multiplication on $\mathcal{S}(M, \mathcal{N})$ and we call say $\mathcal{S}(\Sigma, \mathcal{P})$ is the **stated skein algebra** of (Σ, \mathcal{P})

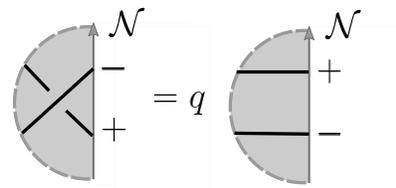
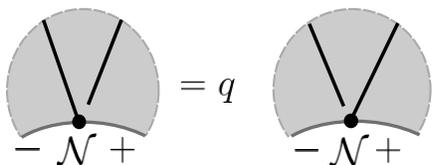


Theorem (Lê 2016)

Diagrams with no crossings, which do not bound disks, and with no negative state “above” a positive state at a marking form a basis of $\mathcal{S}(\Sigma, \mathcal{P})$.

A Quick Note

- This original framework was described for punctured bordered surfaces
- Marked surfaces fit more cleanly into our generalization
- There is a dictionary between the two descriptions similar in spirit to “top view” versus “side view” perspectives



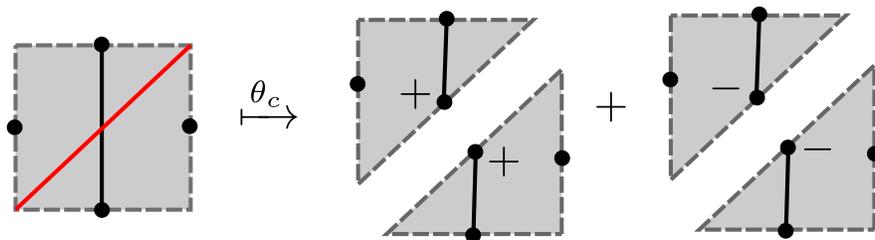
Splitting Along an Ideal Arc

Theorem (Lê 2016)

Let (Σ', \mathcal{P}') be the result of splitting (Σ, \mathcal{P}) along an ideal arc, c . Then there is an algebra embedding

$$\theta_c : \mathcal{S}(\Sigma, \mathcal{P}) \rightarrow \mathcal{S}(\Sigma', \mathcal{P}')$$

Defined by sending a diagram to the sum of all possible equal states on the new endpoints of the split diagram (called lifts).



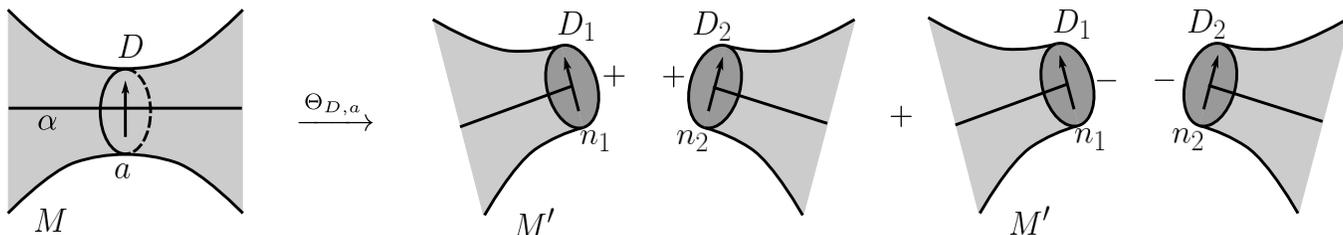
Splitting Along an Embedded Disk

Theorem (Lê and B. 2020)

Let (M', \mathcal{N}') be the result of splitting (M, \mathcal{N}) along a properly embedded disc containing an arc, say (D, a) . Then there is a well defined

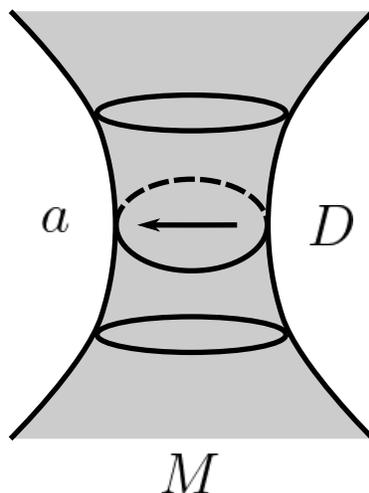
$$\Theta_{D,a} : \mathcal{S}(M, \mathcal{N}) \rightarrow \mathcal{S}(M', \mathcal{N}')$$

Defined by sending \mathcal{N} -tangles passing through a to the sum of all lifts.



Idea of the Proof

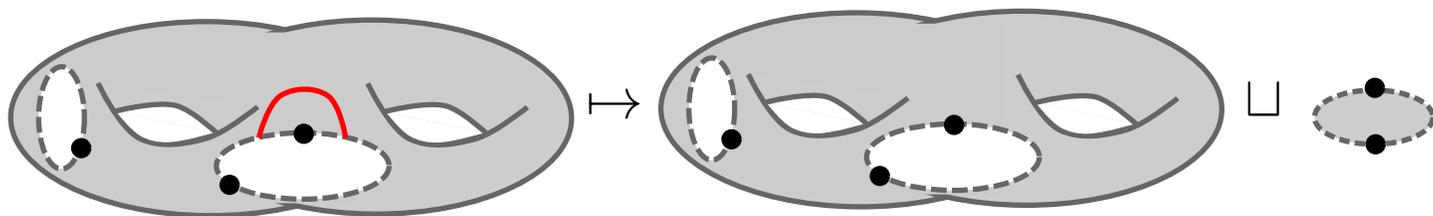
- 1 Take a collared neighborhood of D
- 2 Decompose isotopies and relations into balls of small support
- 3 Reduce to the surface case



A Coaction for Stated Skein Algebras

Theorem (Costantino and Lê 2019)

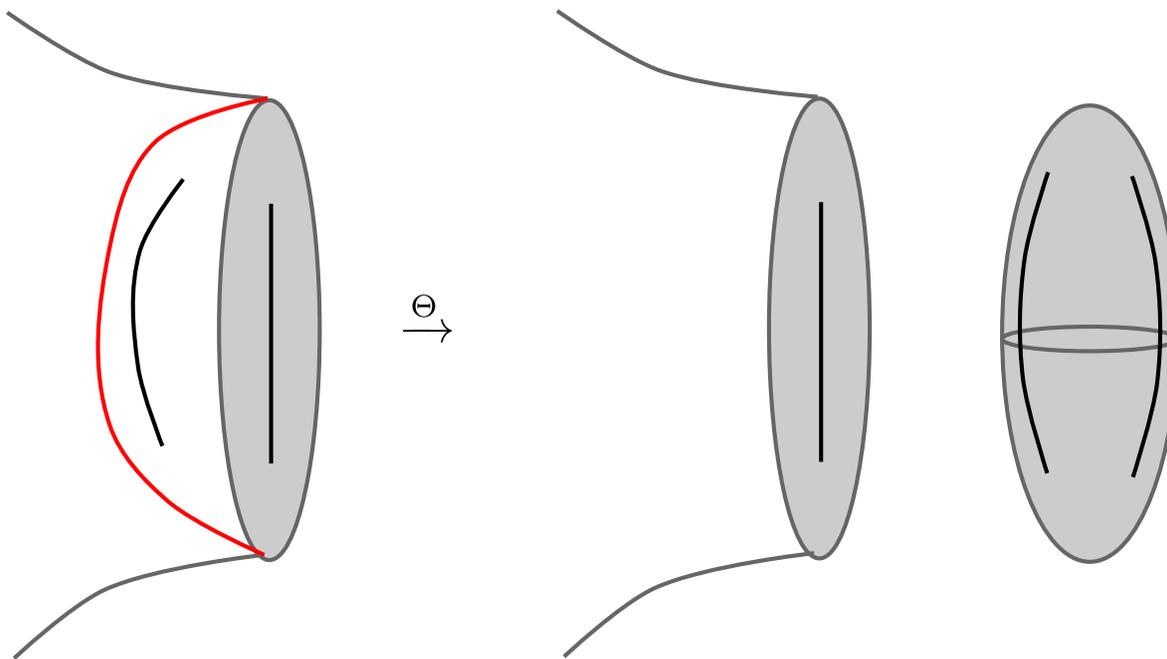
Stated skein algebras of surfaces are comodule algebras over the stated skein algebra of the bigon.



A Coaction for Stated Skein Modules

Let \mathfrak{B} be the skein module of the ball with 2 markings. Then cut along a bubble of a marking to give

$$\mathcal{S}(M, \mathcal{N}) \rightarrow \mathcal{S}(M, \mathcal{N}) \otimes \mathfrak{B}$$



Returning to the Chebyshev Homomorphism

Recall if $\text{ord}(\omega^8) = N$ and $\eta = \omega^{N^2}$ then

$$K_{\eta^2}(\Sigma) \xrightarrow{T_N} K_{\omega^2}(\Sigma)$$

$$\text{knot} \mapsto T_N(\text{knot})$$

- The quasi-classical $K_{\eta^2}(\Sigma)$ admits a nice description in terms of the $SL(2, \mathbb{C})$ character variety of Σ .
- This gives a method for constructing central elements $K_{\omega^2}(\Sigma)$

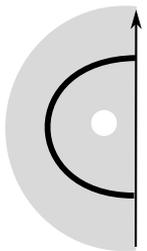
Is there a Chebyshev homomorphism for stated skein modules?

- For Kauffman bracket skein modules of 3–manifolds (Lê 2013)
- For Kauffman bracket skein modules of marked 3–manifolds (Lê-Paprocki 2018)
- A partial result for stated skein algebras (Korinman-Quesney 2019)

Threading

Up to functoriality, every component, α , of a \mathcal{N} -tangle is the core of an annulus or a band (use the framing).

Then $\alpha^{(n)}$ is n parallel copies of this core, and for any polynomial $p(z) = \sum a_i z^i$ we can define $p^{fr}(\alpha) = \sum a_i \alpha^{(i)}$.



α

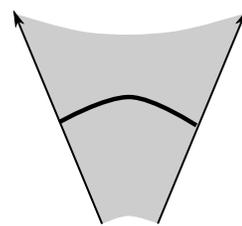


$\alpha^{(N)}$

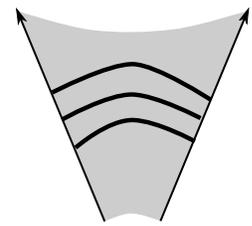
$\mathcal{P}(z)$



α^2



β



$\beta^{(N)}$

$\mathcal{P}(z)$

The Chebyshev-Frobenius Homomorphism

Theorem (Lê and B 2020)

Let ω be a root of unity with $N = \text{ord}(\omega^8)$ and $\eta = \omega^{N^2}$. There is a well defined module homomorphism

$$\Phi_\omega : \mathcal{S}_\eta(M, \mathcal{N}) \rightarrow \mathcal{S}_\omega(M, \mathcal{N})$$

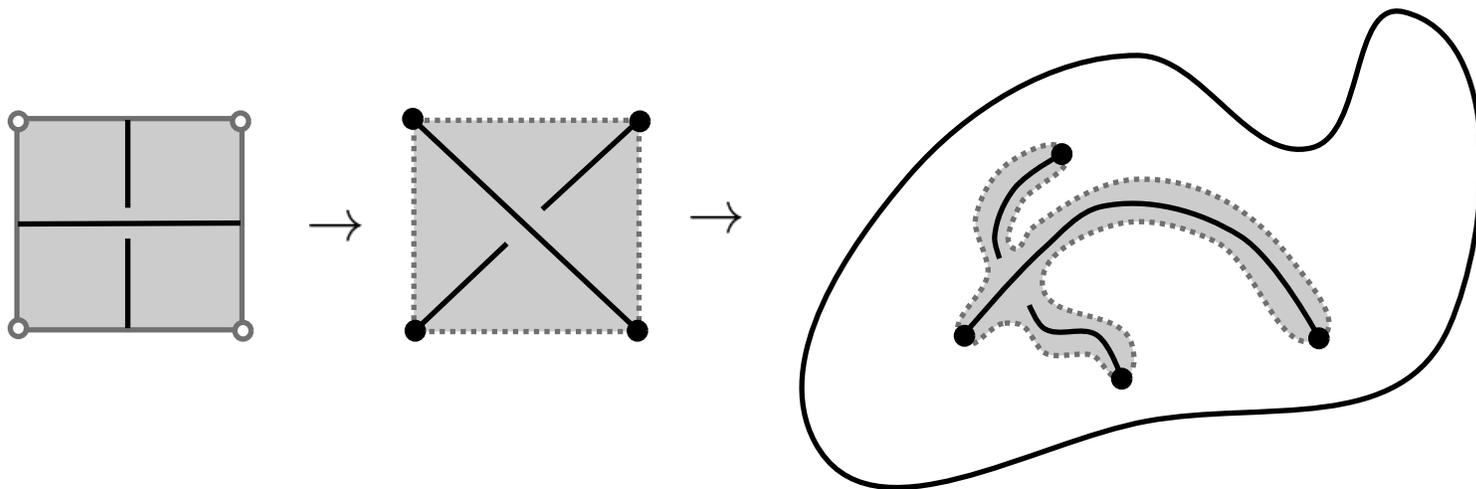
which sends connected link components α to $T_N^{\text{fr}}(\alpha)$ and sends arc components a to $a^{(N)}$.

Moreover, for an embedded disk D

$$\Phi_\omega \circ \Theta_D = \Theta_D \circ \Phi_\omega.$$

Well Defined

- 1 Each relation is of the form $X = Y$. We need $\Phi_\omega(X) = \Phi_\omega(Y)$
- 2 Functoriality gives an “easier” space the relation occurs in
- 3 Prove things hold in the “easy” space and well definedness through (sometimes use the splitting homomorphism is well defined to help).



An Example in the Square

We can set up an induction to deal with the square directly.

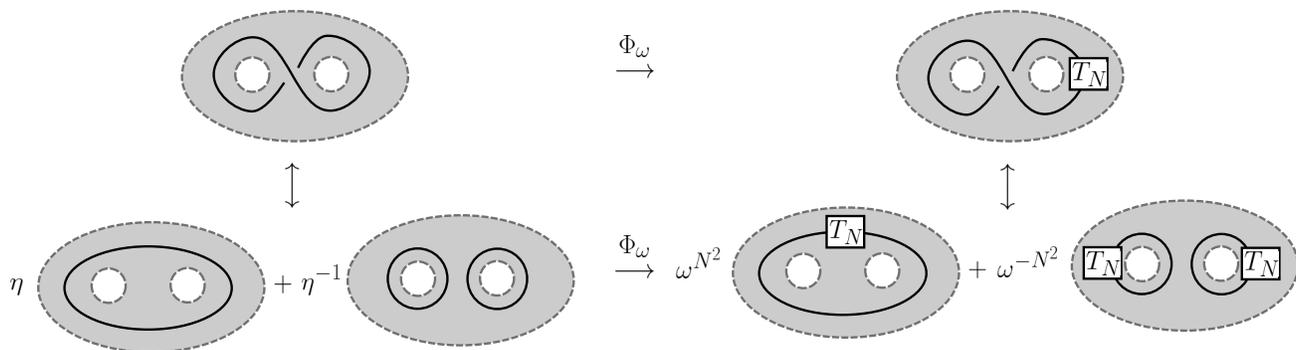
$$\begin{aligned}
 & \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = q \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} + q^{-1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \\
 & = q^{2(2)+1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} + q^{-2(2)-1} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \\
 & = q^5 \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} + q^{-5} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}
 \end{aligned}$$

Specializing to Surfaces

Theorem (Korinman and Quesney 2019)

For surfaces, and $\text{ord}(\omega^8)$ odd, a Chebyshev-Frobenius homomorphism can be defined on the increasingly stated diagram basis.

Our work can be specialized to recover this, and provides extra information.

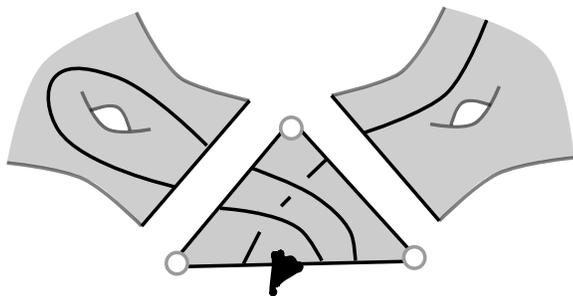


Gluing Along Triangles

Theorem (Costantino and Lê 2019)

Let Σ_1 and Σ_2 be disjoint surfaces, with Σ_3 the result of gluing along a triangle. Then the stated skein algebra of Σ_3 is isomorphic to the “braided tensor product” of the stated skein algebras of Σ_1 and Σ_2

$$\mathcal{S}(\Sigma_3) = \mathcal{S}(\Sigma_1) \underline{\otimes} \mathcal{S}(\Sigma_2)$$



An Observation

Theorem (Lê and B. 2020)

Let $N = \text{ord}(\omega^8)$ and $N' = \text{ord}(\omega^4)$. Then for any α an \mathcal{N} -tangle we have $\Phi_\omega(\alpha)$ is μ -transparent.

In the case of surfaces we see that $T_{2N}(\alpha)$ is central for knots and $a^{(4N)}$ is central for arcs.

The diagram shows an equation between two circular diagrams. Each diagram is a gray circle containing two black lines that cross each other. In the left diagram, the lines cross such that the top-left and bottom-right segments are in the foreground. In the right diagram, the lines cross such that the top-right and bottom-left segments are in the foreground. Both diagrams are labeled with Φ_ω above them. Between the two diagrams is an equals sign followed by the expression $(-1)^{N'+1}$.

$$\Phi_\omega = (-1)^{N'+1} \Phi_\omega$$

An Extension to Surfaces

Corollary (Lê and B 2020)

- *The Chebyshev-Frobenius homomorphism splits over the braided tensor product.*
- *As such the Chebyshev-Frobenius homomorphism of a marked surface is determined by the Chebyshev-Frobenius homomorphism of the bigon.*
- The Chebyshev-Frobenius homomorphism respects splitting and braiding
- A “disk with handles” can be built up from gluing along triangles

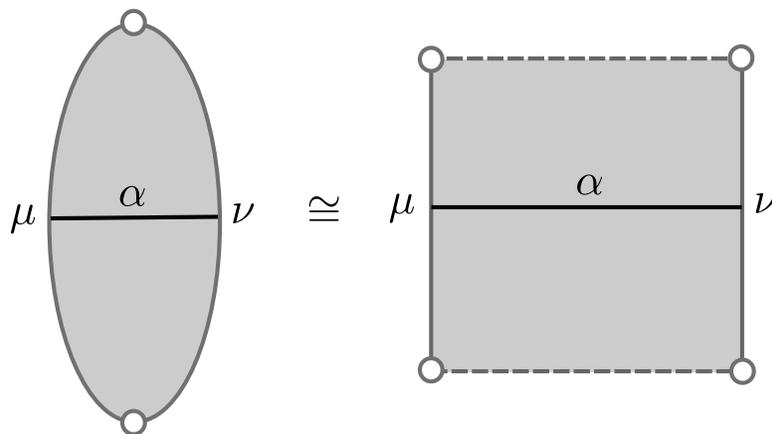
Thank you!

Thank You!

Some of the ignored algebra

But here is a bonus discussion of some of the underlying algebra

The Stated Skein Algebra of the Bigon



Then set

$$\alpha_{++} = a, \quad \alpha_{+-} = b, \quad \alpha_{-+} = c, \quad \alpha_{--} = d$$

Theorem (Costantino and Lê 2019)

This is an isomorphism of the stated skein algebra of the bigon with $\mathcal{O}_{q^2}(SL(2))$.

The Required Quantum Group Definition

Recall $U_{q^2}(\mathfrak{sl}(2))$ is the Hopf algebra over $\mathbb{Q}(q^{1/2})$ generated by $K^{\pm 1}$, E and F with relations

$$KE = q^4 EK, \quad KF = q^{-4} FK, \quad [E, F] = \frac{K - K^{-1}}{q^2 - q^{-2}}$$

with coproduct

$$\Delta(K) = K \otimes K \quad \Delta(E) = 1 \otimes E + E \otimes K \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$

and antipode

$$S(K) = K^{-1} \quad S(E) = -EK^{-1} \quad S(F) = -KF$$

Our Favorite Coordinate Algebra

Define

$$\mathcal{O}(SL(2)) \cong \mathcal{R}[a, b, c, d]/(ad - bc - 1)$$

We have $SL(2)$ is the algebraic group of 2×2 matrices of determinant 1. These are functions that pick out the entries.

$$a : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}$$

so we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Our Favorite Quantum Coordinate Algebra

Define

$$\mathcal{O}_q(SL(2))$$

To be the \mathcal{R} -algebra generated by $a, b, c,$ and d subject to

$$ca = qac, \quad ba = qab, \quad db = qbd, \quad dc = qcd$$

$$ad - q^{-1}bc = 1, \quad da - qcb = 1$$

Hopf Pairing

There is a pairing

$$\langle , \rangle : U_{q^2}(\mathfrak{sl}(2)) \otimes \mathcal{O}_{q^2}(SL(2))$$

defined by

$$\langle K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix},$$

$$\langle E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\langle F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Hopf Duals and Reversing Some Arrows

We have a dictionary:

$$\{\text{right } \mathcal{O}_{q^2}(SL(2)) \text{ - comodule}\} \leftrightarrow \{\text{left } U_{q^2}(\mathfrak{sl}(2)) \text{ - module}\}$$

$$\{M \rightarrow M \otimes \mathcal{O}_{q^2}(SL(2))\} \leftrightarrow \{U_{q^2}(\mathfrak{sl}(2)) \otimes M \rightarrow M\}$$

Lusztig's Frobenius Homomorphism

Let ω be a root of unity, with $N = \text{ord}(\omega^8)$ and $\eta = \omega^{N^2}$. The **Frobenius homomorphism**

$$f : U_{\omega^4}^L(\mathfrak{sl}_2) \rightarrow U_{\eta^4}^L(\mathfrak{sl}_2)$$

is defined by

$$f(K) = (-1)^{N+1} K$$

$$f(E) = 0$$

$$f(F) = 0$$

$$f(E^{(N)}) = E$$

$$f(F^{(N)}) = F$$

A *dual* Frobenius Morphism

If $\text{ord}(q) = N$ is odd, then

$$F_N : \mathcal{O}(SL(2)) \rightarrow \mathcal{O}_q(SL(2))$$

by $a \mapsto a^N$, $b \mapsto b^N$, $c \mapsto c^N$, $d \mapsto d^N$

Notice that the image is in the center!

$$b^N a = q^N a b^N$$

A Skein Theoretic Construction

Theorem

The Chebyshev-Frobenius homomorphism of the bigon is the Hopf Dual of Lusztig's Frobenius homomorphism.

$$\begin{array}{ccc}
 \mathcal{O}_{\eta^4}(SL(2)) \otimes U_{\omega^4}(\mathfrak{sl}_2) & \xrightarrow{\Phi_{\omega} \otimes Id} & \mathcal{O}_{\omega^4}(SL(2)) \otimes U_{\omega^4}(\mathfrak{sl}_2) \\
 \downarrow Id \otimes f & & \downarrow \langle \cdot, \cdot \rangle_{\omega^4} \\
 \mathcal{O}_{\eta^4}(SL(2)) \otimes U_{\eta^4}(\mathfrak{sl}_2) & \xrightarrow{\langle \cdot, \cdot \rangle_{\eta^4}} \mathbb{Z}[\eta^{\pm 4}] & \xleftarrow{\eta = \omega^{N^2}} \mathbb{Z}[\omega^{\pm 4}]
 \end{array}$$

Quantum Tori

Laurent polynomials in variables that q -commute based on a matrix P .

$$\mathbb{T}(P, q) := \mathcal{R}\langle \{x_i\}_{i=1}^{|P|} : x_i x_j = q^{P_{ij}} x_j x_i \rangle$$

Given any integer N there is a **Frobenius homomorphism**

$$F_N : \mathbb{T}(P, q^{N^2}) \rightarrow \mathbb{T}(P, q), \quad x_i \mapsto x_i^N$$

A Quantum Trace Map

Theorem (Lê and Yu 2020)

The stated skein algebra of a (quasitriangular) marked surface (Σ, \mathcal{P}) embeds into a quantum torus

$$\mathcal{S}_q(\Sigma, \mathcal{P}) \hookrightarrow \mathbb{T}(P_{\mathcal{E}}, q),$$

Where $P_{\mathcal{E}}$ depends on a quasitriangulation \mathcal{E} of (Σ, \mathcal{P}) .

Idea: Find enough q -commuting elements.

$$P(a, b) = \# \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) - \# \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The diagrams are:
1. A shaded sector with vertex p and two rays labeled b and a .
2. A shaded sector with vertex p and two rays labeled a and b .

A Natural Question

$$\begin{array}{ccc} \mathcal{S}_{\omega^{N^2}}(\Sigma, \mathcal{P}) & \hookrightarrow & \mathbb{T}(P_{\mathcal{E}}, \omega^{N^2}) \\ \downarrow \text{?} & & \downarrow F_N \\ \mathcal{S}_{\omega}(\Sigma, \mathcal{P}) & \hookrightarrow & \mathbb{T}(P_{\mathcal{E}}, \omega) \end{array}$$

An Answer

Theorem (Lê and B 2020)

Suppose (Σ, \mathcal{P}) has at least two quasitriangulations, then F_N restricts to a map of skein algebras and the restriction does not depend on the quasitriangulation if and only if ω is a root of unity with $N = \text{ord}(\omega^8)$. In this case, the restriction is Φ_ω .

$$\begin{array}{ccc} \mathcal{S}_{\omega^{N^2}}(\Sigma, \mathcal{P}) & \hookrightarrow & \mathbb{T}(P_{\mathcal{E}}, \omega^{N^2}) \\ \downarrow \Phi_\omega & & \downarrow F_N \\ \mathcal{S}_\omega(\Sigma, \mathcal{P}) & \hookrightarrow & \mathbb{T}(P_{\mathcal{E}}, \omega) \end{array}$$