Dynamics on character varieties and geometric structures

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University of Maryland

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George Washington University

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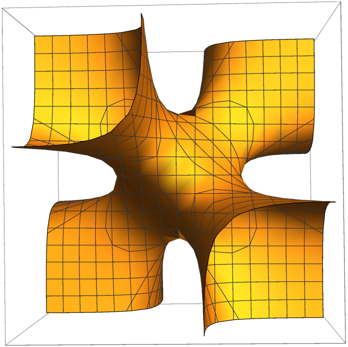
Abstract

Abstract: The classification of geometries on low-dimensional manifolds leads to interesting dynamical systems on moduli spaces associated with surfaces. In my talk I survey some examples and describe some future directions.



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  - Affine structures (flat affine connections of zero torsion).

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    - $\mathrm{Def}_{(G,X)}(\Sigma)$ itself locally modeled on $\mathrm{Rep}(\pi, G)$
    - $\mathrm{Mod}(\Sigma)$-action on $\mathrm{Def}_{(G,X)}(\Sigma)$ corresponds to $\mathrm{Out}(\pi)$-action on $\mathrm{Rep}(\pi, G)$.

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  - Modular group $\mathsf{Mod}(\Sigma) \cong \mathsf{GL}(2,\mathbb{Z})$ acts *properly* by linear fractional transformations on H^2.

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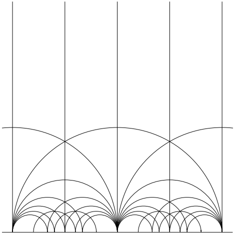

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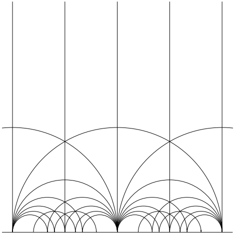

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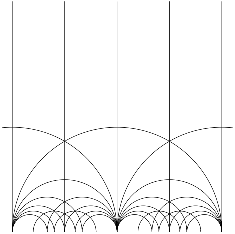

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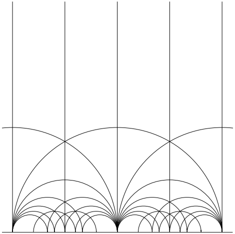

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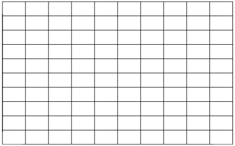
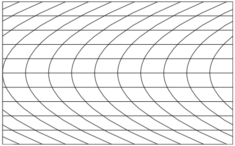
- If $\chi(S) < 0$, Choi-G implies $\mathrm{Mod}(\Sigma)$ acts properly on $\mathbb{R}\mathrm{P}^2(S)$.
- In contrast, *complete affine* structures on $S = T^2$ define $\mathbb{R}^2 \subset \mathbb{R}\mathrm{P}^2(S)$ with usual *linear action* of $\mathrm{GL}(2, \mathbb{Z})$. (O. Baues 2000).

Complete affine surfaces





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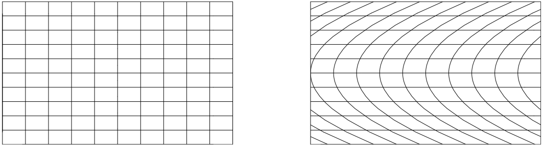

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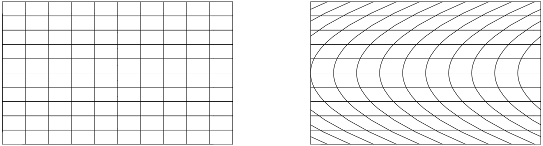

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$$\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$$
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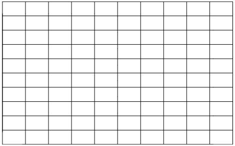

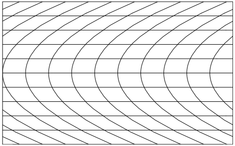

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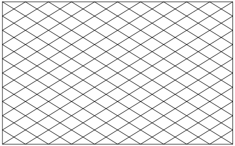
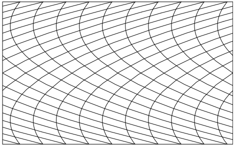
- If translation $\lambda(x, y) = (x + s, y + t)$ lies in the lattice Λ, then

$$\Big(x, y\Big) \xrightarrow{\phi\lambda\phi^{-1}} \Big(x + 2ty + (s + t^2), y + t\Big)$$

is *affine*.

Chaotic dynamics





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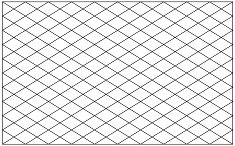

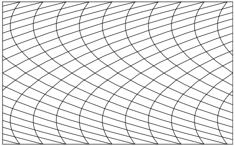

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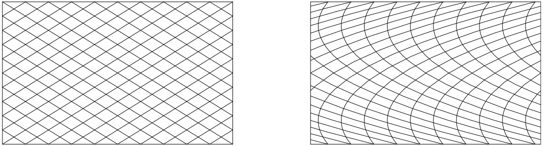

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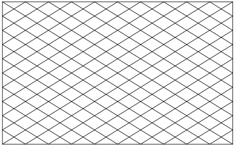

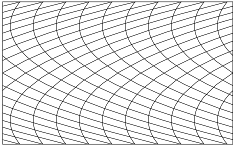

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  - (Moore 1966) Action is *ergodic:*

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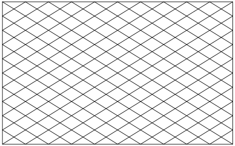

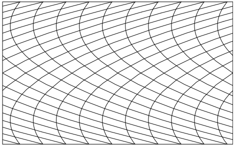

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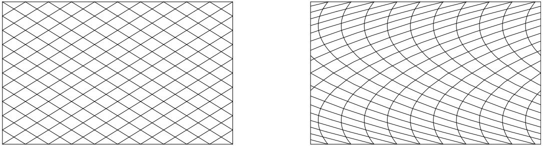

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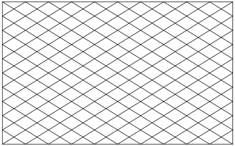

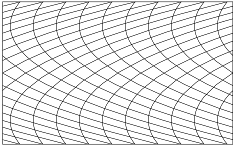

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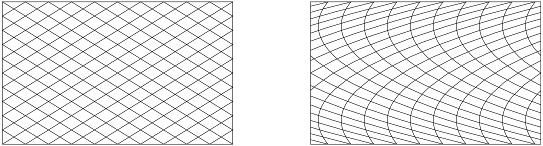

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- Therefore, the classification of geometric structures should be more insightfully regarded as a *dynamical system,* since the moduli space — its quotient — is often intractable.

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- When $\partial\Sigma \neq \emptyset$, then $\mathrm{Rep}(\pi, G)$ inherits a *Poisson structure.* Imposing boundary conditions (for example, fixing the conjugacy classes of the boundary holonomy) defines a map

$$\mathrm{Rep}(\pi, G) \longrightarrow \mathrm{Rep}\big(\pi_1(\partial\Sigma), G\big)$$

whose fibers have symplectic structures.

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- Main technique for proving ergodicity uses dynamics of Dehn twists in $\mathrm{Mod}(\Sigma)$.

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$$\begin{aligned} \mathrm{Rep}(\pi, G) &\xrightarrow{f_\gamma} \mathbb{R} \\ [\rho] &\mapsto \Re\big(\mathrm{Tr}\rho(\gamma)\big) \end{aligned}$$

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- For $G = \mathsf{SL}(2)$, character functions f_γ of simple γ generate coordinate ring of $\mathsf{Rep}(\pi, G)$.

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- The $\mathrm{Inn}\big(\mathrm{SL}(2)\big)$-invariant mapping

$$\begin{aligned} \mathrm{Hom}(F_2, \mathrm{SL}(2)) &\longrightarrow \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} \xi := & \mathrm{Tr}\big(\rho(X)\big) \\ \eta := & \mathrm{Tr}\big(\rho(Y)\big) \\ \zeta := & \mathrm{Tr}\big(\rho(XY)\big) \end{bmatrix} \end{aligned}$$

  defines an isomorphism

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- Out(F_2)-invariant commutator trace function:

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  - Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of F_2.

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  - Weil-Petersson symplectic structure on Fricke spaces $k \leq -2$;
  - Narasimhan-Atiyah-Bott structure for $G = \mathsf{SU}(2)$.
- $\kappa^{-1}(k)$ are the *relative character varieties.*

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- Nonlinear automorphisms of

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generated by *involutions:*

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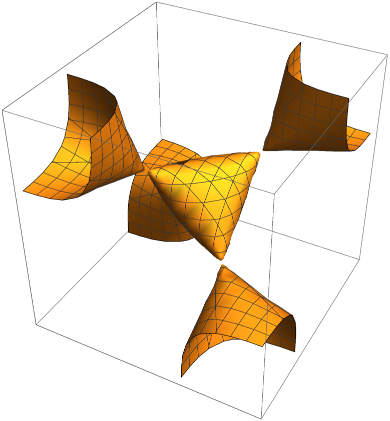
  - Fixing η and ζ yields quadratic equation in ξ;

$$\xi^2 - \ (\eta\zeta)\ \xi \ = \ k + 2 - \eta^2 - \zeta^2$$

    whose roots ξ and $\xi' = \eta\zeta - \xi$ sum to linear coefficient $\eta\zeta$.

Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$



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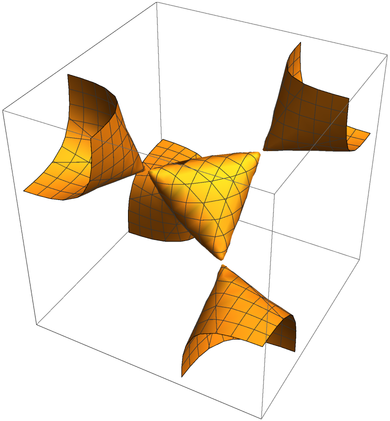

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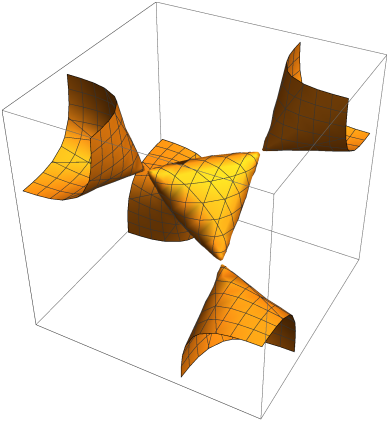
$$\xi = a + a^{-1}, \qquad \eta = b + b^{-1}, \qquad \zeta = ab + (ab)^{-1}$$



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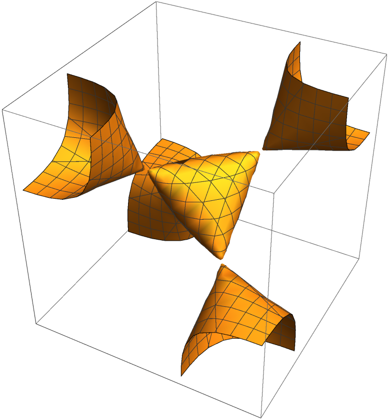
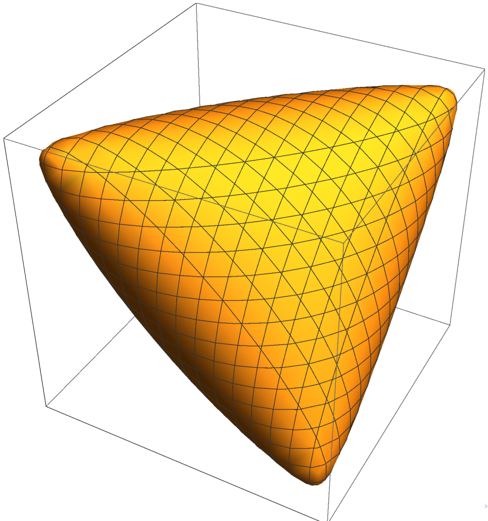
  - *Homogeneous dynamics:* $\mathrm{GL}(2, \mathbb{Z})$-action on $(\mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{Z}/2)$.



$\mathbb{R}$-points: Unitary representations



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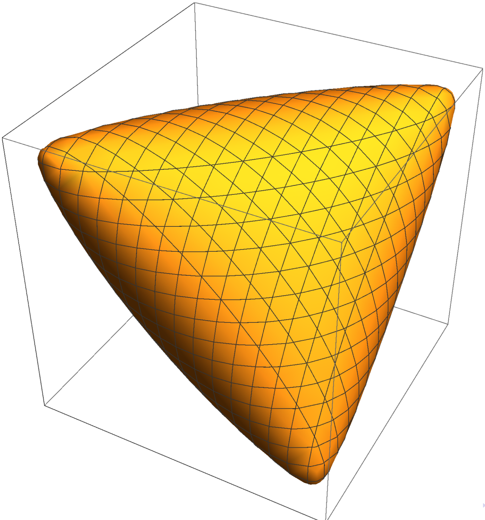

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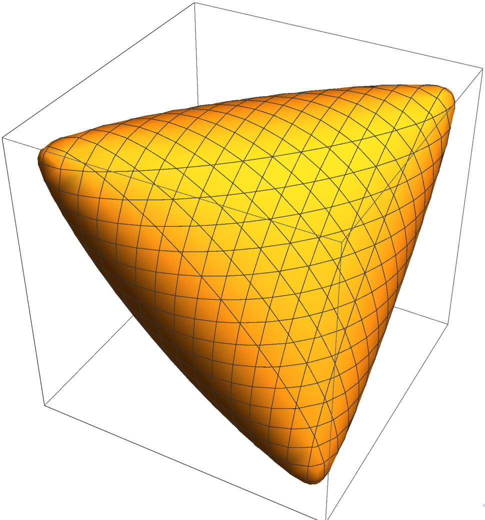
- Characters in $[-2, 2]^3$ with $\kappa \leq 2 \longleftrightarrow$ SU(2)-representations.



$\mathbb{R}$-points: Hyperbolic structures on orientable surfaces

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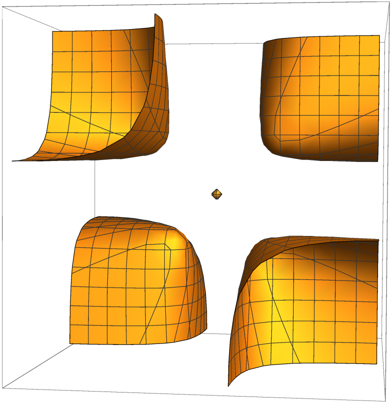
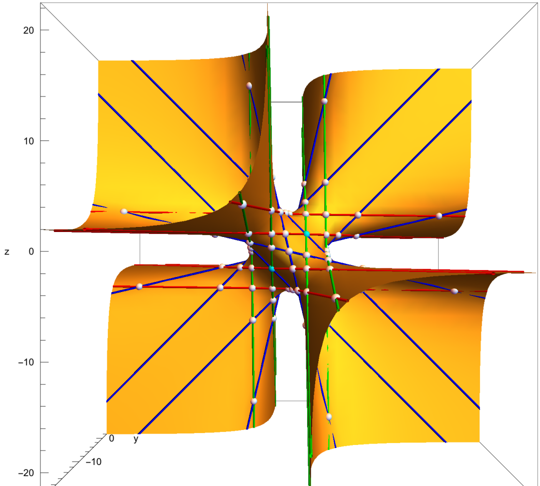
  - Homotopy-equivalences $\Sigma_{1,1} \rightsquigarrow \Sigma_{0,3}$ (and other surfaces with $\pi_1 \cong F_2$) form wandering domains for Out(F_2)-action.

Example: The Markoff surface $x^2 + y^2 + z^2 = xyz$



$\mathbb{R}^3 \cap \kappa^{-1}(-2)$ parametrizes hyperbolic structures on the punctured torus. The origin $(0, 0, 0)$ corresponds to the unique SU(2)-representation with $k = -2$. The famous *Markoff triples* correspond to triply symmetric hyperbolic punctured tori.

Fricke orbits define wandering domains for $k > 2$



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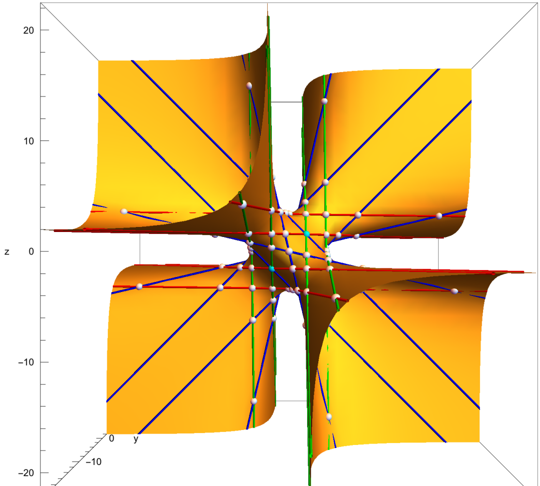

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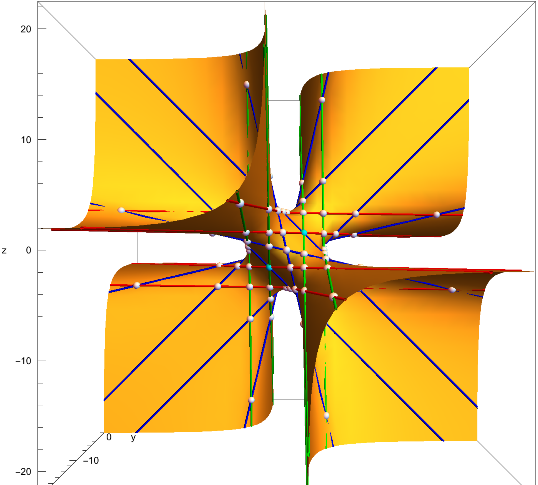

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- For $k > 18$, action ergodic on complement.