The skein algebra of the 4-punctured sphere from curve counting

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Introduction

- Talk based on the recent preprint arXiv:2009.02266.

- Main result: proof of a conjecture in the paper "On multiplying curves in the Kauffman bracket skein algebra of the thickened four-holed sphere" (Bakshi, Mukherjee, Przytycki, Silvero, Wang, 2018) on the positivity of some basis of the Kauffman bracket skein algebra of the 4-punctured sphere.

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- Skein algebras of topological surfaces.

- Conjectural positive bases of skein algebras.

- Results for the skein algebra of the 4-punctured sphere (and of the 1-punctured torus).

- Algorithm: scattering diagrams and broken lines for the skein algebra of the 4-punctured sphere.

- The "quantum mirror" of the cubic surface (computations in higher genus log Gromov-Witten theory).

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Skein modules of 3-manifolds

- The Kauffman bracket skein module (Przytycki, Turaev, 1988) of an oriented 3-manifold $\mathbb{M}$ is the $\mathbb{Z}[A^{\pm}]$-module generated by isotopy classes of framed links in $\mathbb{M}$ satisfying the skein relations

$$\asymp = A \asymp + A^{-1} \rangle\langle \quad \text{and} \quad L \cup \bigcirc = -(A^2 + A^{-2})\, L.$$

- The diagrams in each relation indicate framed links that can be isotoped to identical embeddings except within the neighborhood shown, where the framing is vertical.

- The skein module of $\mathbb{M} = \mathbb{R}^3$ is $\mathbb{Z}[A^{\pm}]$ (generated by the empty link). The class of a framed link $L \subset \mathbb{R}^3$ in $\mathbb{Z}[A^{\pm}]$ is the Kauffman bracket polynomial of L (equivalent to the Jones polynomial).

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Skein algebras of surfaces

- Given an oriented 2-manifold $\mathbb{S}$, one can define a natural algebra structure on the Kauffmann bracket skein module of the 3-manifold $\mathbb{M} := \mathbb{S} \times (-1, 1)$: given two framed links L_1 and L_2 in $\mathbb{S} \times (-1, 1)$, and viewing the interval $(-1, 1)$ as a vertical direction, the product $L_1 L_2$ is defined by placing L_1 on top of L_2.

- We denote by $\mathrm{Sk}_A(\mathbb{S})$ the resulting associative $\mathbb{Z}[A^{\pm}]$-algebra with unit. The skein algebra $\mathrm{Sk}_A(\mathbb{S})$ is in general non-commutative.

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Skein algebras of surfaces

- We consider the case where $\mathbb{S}$ is the complement $\mathbb{S}_{g,\ell}$ of a finite number ℓ of points in a compact oriented 2-manifold of genus g.

- A multicurve on $\mathbb{S}_{g,\ell}$ is the union of finitely many disjoint compact connected embedded 1-dimensional submanifolds of $\mathbb{S}_{g,\ell}$ such that none of them bounds a disc in $\mathbb{S}_{g,\ell}$. Identifying $\mathbb{S}_{g,\ell}$ with $\mathbb{S}_{g,\ell} \times \{0\} \subset \mathbb{S}_{g,\ell} \times (-1,1)$, a multicurve on $\mathbb{S}_{g,\ell}$ endowed with the vertical framing naturally defined a framed link in $\mathbb{S}_{g,\ell} \times (-1,1)$.

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Isotopy classes of multicurves form a basis of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\mathbb{Z}[A^{\pm}]$-module.

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The skein algebra as deformation quantization of the SL_2 character variety

- $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$: the SL_2-character variety of the $\mathbb{S}_{g,\ell}$, affine variety of finite type over $\mathbb{Z}$ obtained as affine GIT quotient by the SL_2 conjugation action of the affine variety of representations of the fundamental group $\pi_1(\mathbb{S}_{g,\ell})$ into SL_2.

- For every γ multicurve on $\mathbb{S}_{g,\ell}$ with connected components $\gamma_1, \cdots, \gamma_r$, the map sending a representation $\rho\colon \pi_1(\mathbb{S}_{g,\ell}) \to SL_2$ to $\prod_{j=1}^r (-\mathrm{tr}(\rho(\gamma_j)))$ defines a regular function f_γ on $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$.

Theorem (Bullock, Przytycki-Sikora, Charles-Marché)

The map $\gamma \mapsto f_\gamma$ defines a ring isomorphism between the specialization $\mathrm{Sk}_{-1}(\mathbb{S}_{g,\ell})$ of the skein algebra at $A = -1$ and the ring of regular functions of $\mathrm{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$.

Classical limit of the skein relation: for every $M, N \in SL_2$,

$$\mathrm{tr}(M)\,\mathrm{tr}(N) = \mathrm{tr}(MN) + \mathrm{tr}(M^{-1}N)\,.$$

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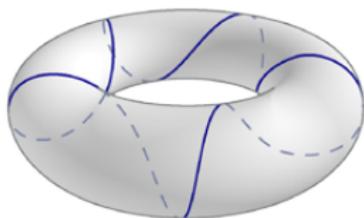
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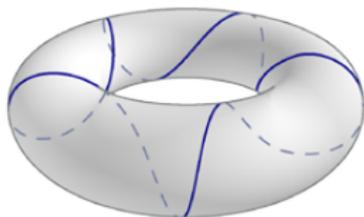$$\operatorname{tr}(M)\operatorname{tr}(N) = \operatorname{tr}(MN) + \operatorname{tr}(M^{-1}N).$$

Example: closed torus

- On the closed torus $\mathbb{S}_{1,0}$, isotopy classes of multicurves are in bijection with

  $$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\}.$$
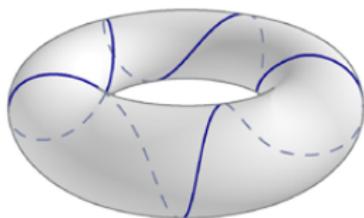
- For every $p = (m, n) \in B(\mathbb{Z})$, denote by γ_p the corresponding isotopy class of multicurves.

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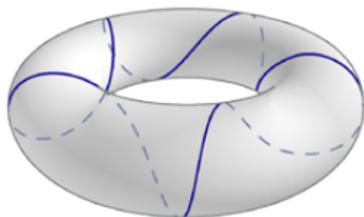
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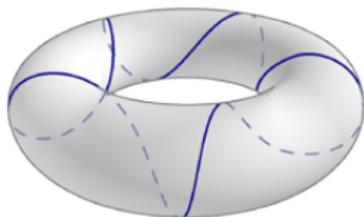
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Structure constants and positive bases

- The structure constants $C^l_{j,k} \in R$ of a basis $\{e_j\}_{j \in J}$ of an algebra $\mathcal{A}$ over a ring R are defined by

$$e_j e_k = \sum_{l \in J} C^l_{j,k} e_l \,.$$

- For the skein algebra, $R = \mathbb{Z}[A^\pm]$.

Definition

A basis $\{e_j\}_{j \in J}$ of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is called *positive* if its structure constants belong to $\mathbb{Z}_{\geq 0}[A^\pm]$, i.e. are Laurent polynomials in A with positive coefficients.

- Question: is the basis of multicurves positive?

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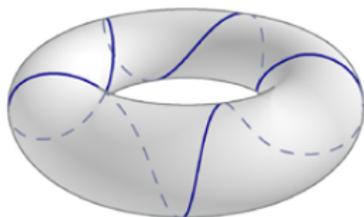
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- One crossing to resolve:

$$\gamma_{(1,0)}\gamma_{(0,1)} = A\gamma_{1,1} + A^{-1}\gamma_{(-1,1)}.$$

- Two crossings to resolve:

$$\gamma_{(0,1)}\gamma_{(2,1)} = A^{-2}\gamma_{(2,2)} + A^2\gamma_{(2,0)} - 2A^{-2} - 2A^2.$$

- Conclusion: the basis of multicurves is not positive in general.
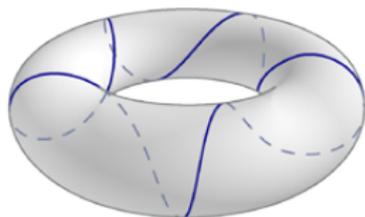
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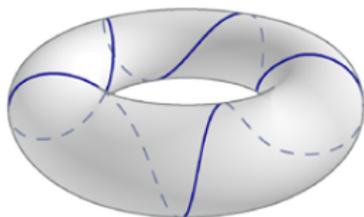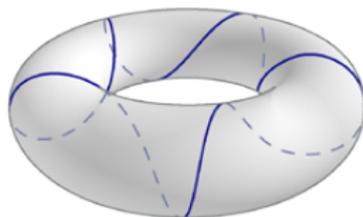
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The bracelets basis

- Let $T_n(x)$ be the Chebyshev polynomials defined by

$$T_0(x) = 1, \, T_1(x) = x, \, T_2(x) = x^2 - 2,$$

and for every $n \geq 2$,

$$T_{n+1}(x) = x T_n(x) - T_{n-1}(x).$$

Writing $x = \lambda + \lambda^{-1}$, we have $T_n(x) = \lambda^n + \lambda^{-n}$ for every $n \geq 1$.

- Given an isotopy class γ of multicurve on $\mathbb{S}_{g,\ell}$, one can uniquely write γ in $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$ where $\gamma_1, \cdots, \gamma_r$ are all distinct isotopy classes of connected multicurves and $n_j \in \mathbb{Z}_{>0}$, and we define

$$\mathbf{T}(\gamma) := T_{n_1}(\gamma_1) \cdots T_{n_r}(\gamma_r).$$

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$$T_{n+1}(x) = x T_n(x) - T_{n-1}(x).$$

Writing $x = \lambda + \lambda^{-1}$, we have $T_n(x) = \lambda^n + \lambda^{-n}$ for every $n \geq 1$.

- Given an isotopy class γ of multicurve on $\mathbb{S}_{g,\ell}$, one can uniquely write γ in $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$ where $\gamma_1, \cdots, \gamma_r$ are all distinct isotopy classes of connected multicurves and $n_j \in \mathbb{Z}_{>0}$, and we define

$$\mathbf{T}(\gamma) := T_{n_1}(\gamma_1) \cdots T_{n_r}(\gamma_r).$$

- $\{\mathbf{T}(\gamma)\}_\gamma$: *bracelets basis* of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$.

The bracelets basis

- If γ is a connected multicurve on $\mathbb{S}_{g,\ell}$, then $T_n(\gamma)$ is a quantizations of the function $\rho \mapsto -\operatorname{tr}(\rho(\gamma)^n)$ on $\operatorname{Ch}_{SL_2}(\mathbb{S}_{g,\ell})$.

- Any hope of positivity?

$$T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x).$$

$$(\lambda^m + \lambda^{-m})(\lambda^n + \lambda^{-n}) = (\lambda^{m+n} + \lambda^{-m-n}) + (\lambda^{m-n} + \lambda^{n-m}).$$

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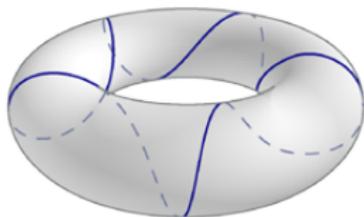
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Example: closed torus

- As $T_1(x) = x$ and $T_2(x) = x^2 - 2$,

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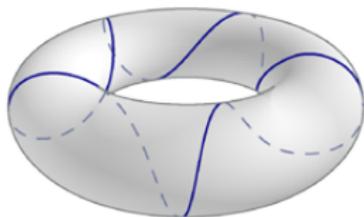
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Conjectural positivity of the bracelets basis

Conjecture (Dylan Thurston, 2013)

For every g and ℓ, the bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is positive.

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For every g and ℓ, after setting $A = 1$, the structure constants of the bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ are non-negative.

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The bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{1,0})$ of the closed torus $\mathbb{S}_{1,0}$ is positive. In fact, for every $p_1, p_2 \in B(\mathbb{Z})$,

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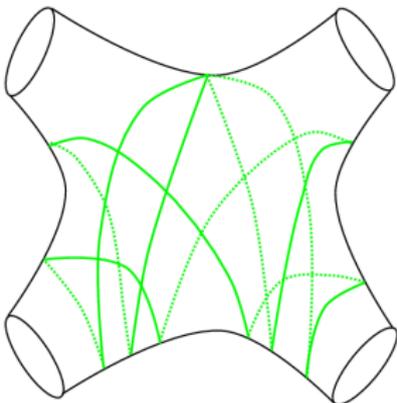
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Theorem (B, 2020)

The bracelets bases $\{\mathbf{T}(\gamma)\}_\gamma$ of the skein algebras $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$ of the 4-punctured sphere and the 1-punctured torus are positive.

Unlike the case of the closed torus $\mathbb{S}_{1,0}$, there does not seem to exist a simple closed formula for the structure constants of the bracelets basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$.

Main result

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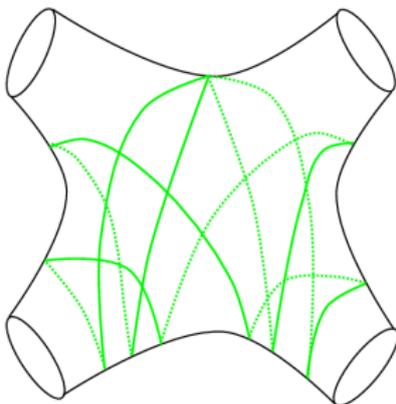
- Focus on the case of the 4-punctured sphere $\mathbb{S}_{0,4}$.
- Peripheral curves a_1, a_2, a_3, a_4, in the center of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$, so we can view $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as a R-module, where $R = \mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$.
- Isotopy classes of multicurves in $\mathbb{S}_{0,4}$ without peripheral connected components are in bijection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \ \text{if} \ n = 0\}.$$

(View $\mathbb{S}_{0,4}$ as a $\mathbb{Z}/2\mathbb{Z}$-quotient of a 4-punctured torus)

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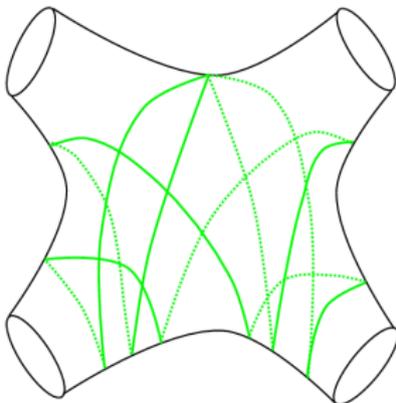
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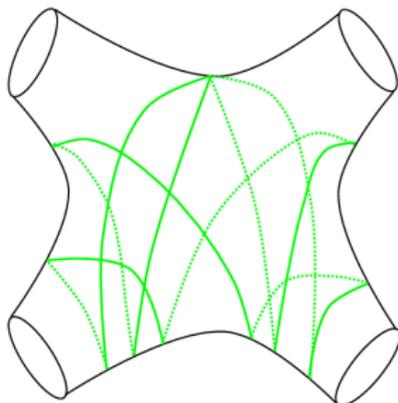- $\{\gamma_p\}_{p \in B(\mathbb{Z})}$ is a basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as R-module.

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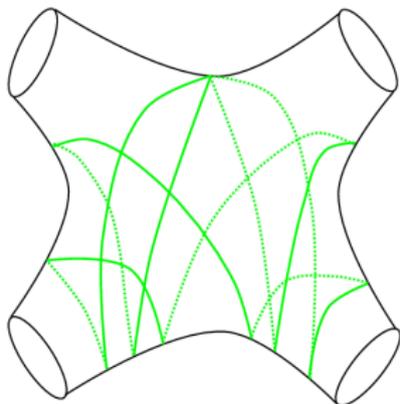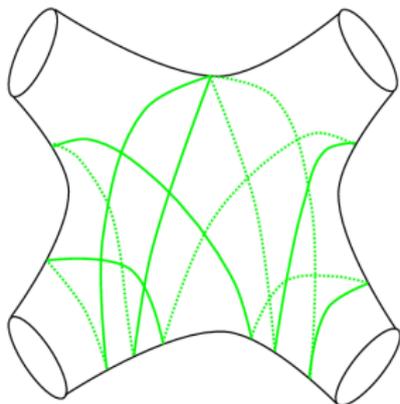- $\{\gamma_p\}_{p \in B(\mathbb{Z})}$ is a basis of $Sk_A(\mathbb{S}_{0,4})$ as R-module.

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Example

- $$\mathbf{T}(\gamma_{(1,0)})\mathbf{T}(\gamma_{(0,1)}) = A^2\mathbf{T}(\gamma_{(1,1)}) + A^{-2}\mathbf{T}(\gamma_{(-1,1)}) + R_{1,1}\,.$$

- $$\mathbf{T}(\gamma_{(1,1)})\mathbf{T}(\gamma_{(-1,1)}) = A^{-4}\mathbf{T}(\gamma_{(2,0)}) + A^4\mathbf{T}(\gamma_{(0,2)}) + A^{-2}R_{1,0} + A^2R_{0,1} + y\,.$$

Generators and relations

Theorem (Bullock-Przytycki, 2000)

$\mathrm{Sk}_A(\mathbb{S}_{0,4})$ is generated as R-algebra by $\gamma_{v_1} := \gamma_{(1,0)}$, $\gamma_{v_2} := \gamma_{(0,1)}$, $\gamma_{v_3} = \gamma_{(-1,1)}$, with the relations

$$A^{-2}\gamma_{v_1}\gamma_{v_2} - A^2\gamma_{v_2}\gamma_{v_1} = (A^{-4} - A^4)\gamma_{v_3} - (A^2 - A^{-2})R_{1,1}\,,$$

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$$A^{-2}\gamma_{v_1}\gamma_{v_2}\gamma_{v_3} = A^{-4}\gamma_{v_1}^2 + A^4\gamma_{v_2}^2 + A^{-4}\gamma_{v_3}^2 + A^{-2}R_{1,0}\gamma_{v_1} + A^2R_{0,1}\gamma_{v_2}$$

$$+A^{-2}R_{1,1}\gamma_{v_3} + y - 2(A^4 + A^{-4})\,.$$

Generators and relations

Setting $A = -1$, we get that the ring of regular fuctions on the SL_2 character variety $\mathsf{Ch}_{SL_2}(\mathbb{S}_{0,4})$ is generated as $\mathbb{Z}[a_1, a_2, a_3, a_4]$-algebra by $\gamma_{v_1} := \gamma_{(1,0)}$, $\gamma_{v_2} := \gamma_{(0,1)}$, $\gamma_{v_3} = \gamma_{(-1,1)}$, with the cubic relation

$$\gamma_{v_1} \gamma_{v_2} \gamma_{v_3} = \gamma_{v_1}^2 + \gamma_{v_2}^2 + \gamma_{v_3}^2 + R_{1,0}\gamma_{v_1} + R_{0,1}\gamma_{v_2} + R_{1,1}\gamma_{v_3} + y - 4\,.$$

In other words, $\mathsf{Ch}_{SL_2}(\mathbb{S}_{0,4})$ is a 4-parameters families of affine cubic surfaces (complement of a triangle of lines in a projective cubic surface)(known at the end of the 19th century: Vogt, Fricke-Klein). Think about the skein algebra $\mathsf{Sk}_A(\mathbb{S}_{0,4})$ as a family of non-commutative cubic surfaces.

$PSL_2(\mathbb{Z})$ action

- The mapping class group of $\mathbb{S}_{g,\ell}$ (group of isotopy classes of orientation-preserving diffeomorphisms) acts on $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$. As the mapping class group of $\mathbb{S}_{0,4}$ contains a copy of $PSL_2(\mathbb{Z})$, we get an action on $PSL_2(\mathbb{Z})$ on $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ by R-algebra automorphisms.

- For every $p \in B(\mathbb{Z}) = \mathbb{Z}^2/\langle \pm id \rangle$ and $M \in PSL_2(\mathbb{Z})$, we have $M.\gamma_p = \gamma_{M.p}$.

- Generators of $PSL_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- Action of S and T on generators of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$:

$$S\gamma_{v_1} = \gamma_{v_2}, S\gamma_{v_2} = \gamma_{v_3}, S\gamma_{v_3} = \gamma_{v_1},$$

$$T\gamma_{v_1} = \gamma_{v_1}, T\gamma_{v_2} = A^{-2}\gamma_{v_1}\gamma_{v_2} - A^{-4}\gamma_{v_3} - A^{-2}R_{1,1}, T\gamma_{v_3} = \gamma_{v_2}.$$

$PSL_2(\mathbb{Z})$ action

- The mapping class group of $\mathbb{S}_{g,\ell}$ (group of isotopy classes of orientation-preserving diffeomorphisms) acts on $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$. As the mapping class group of $\mathbb{S}_{0,4}$ contains a copy of $PSL_2(\mathbb{Z})$, we get an action on $PSL_2(\mathbb{Z})$ on $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ by R-algebra automorphisms.

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Structure of the proof

- Algorithm computing the structure constants $C_{p_1,p_2}^p \in R$ defined by

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and making manifest their positivity properties.

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Scattering diagram

We have $B(\mathbb{Z}) \subset B$, where

$$B := \mathbb{R}^2 / \langle \pm id \rangle \simeq \{(x, y) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid x \geq 0 \text{ if } y = 0\}.$$

"Scattering diagram": attach a power series to every ray in B with rational slope of primitive direction $(m, n) \in B(\mathbb{Z})$

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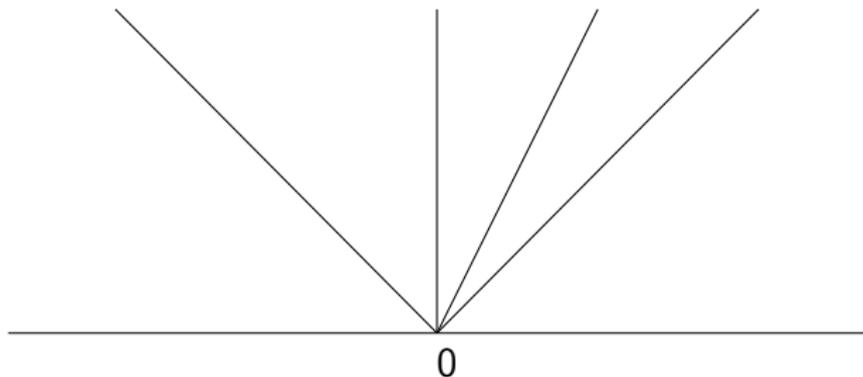$$f_{m,n} = \sum_{k \geq 0} c_k z^{-(km, kn)} \in R[[z^{-(m,n)}]].$$

Broken lines

- γ broken line in $\mathfrak{D}$ of asymptotic direction $p \in B(\mathbb{Z})$ and endpoint Q

- Continuous piecewise integral affine line, bending along rays of rational slopes, decorated by monomials.

- Monomial attached to the linearity domain L of the form $c_L z^{p_L}$, where $c_L \in R$, and $-p_L \in \mathbb{Z}^2$ parallel to the direction of L.

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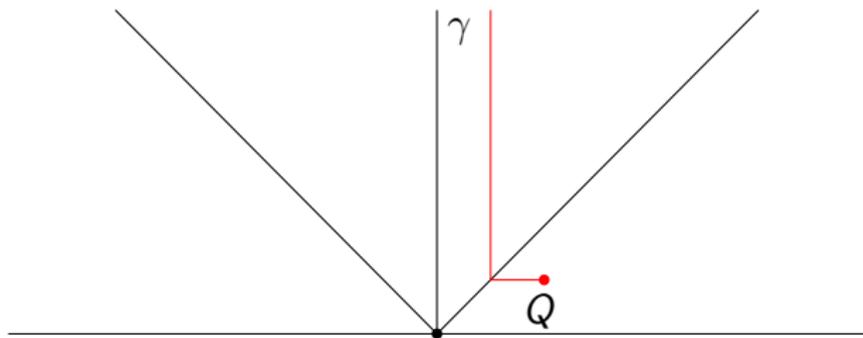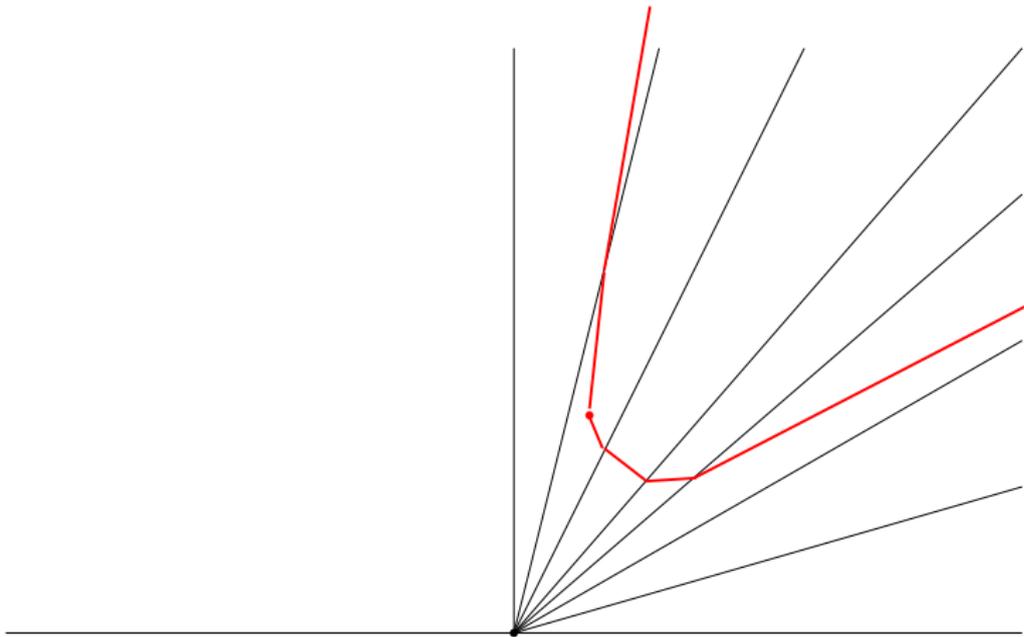- Asymptotic line parallel to p, with monomial z^{-p}.

Broken lines

- Bending formula for a broken line passing between the domains of linearity L and L' by bending along $\mathbb{R}_{\geq 0}(m, n)$.

- Write $m_L = c_L z^{p_L}$, $m_{L'} = c_{L'} z^{p_{L'}}$, $N = |\det((m, n), p_L)|$, and $f_{m,n} = \sum_{k \geq 0} c_k z^{-(km, kn)}$, then there exists a monomial $d_\ell z^{-\ell(m,n)}$ in the series

$$\sum_{\ell \geq 0} d_\ell z^{-(\ell m, \ell n)} := \prod_{j=0}^{N-1} \left(\sum_{k \geq 0} c_k A^{4k(j - \frac{N-1}{2})} z^{-k(m,n)} \right),$$

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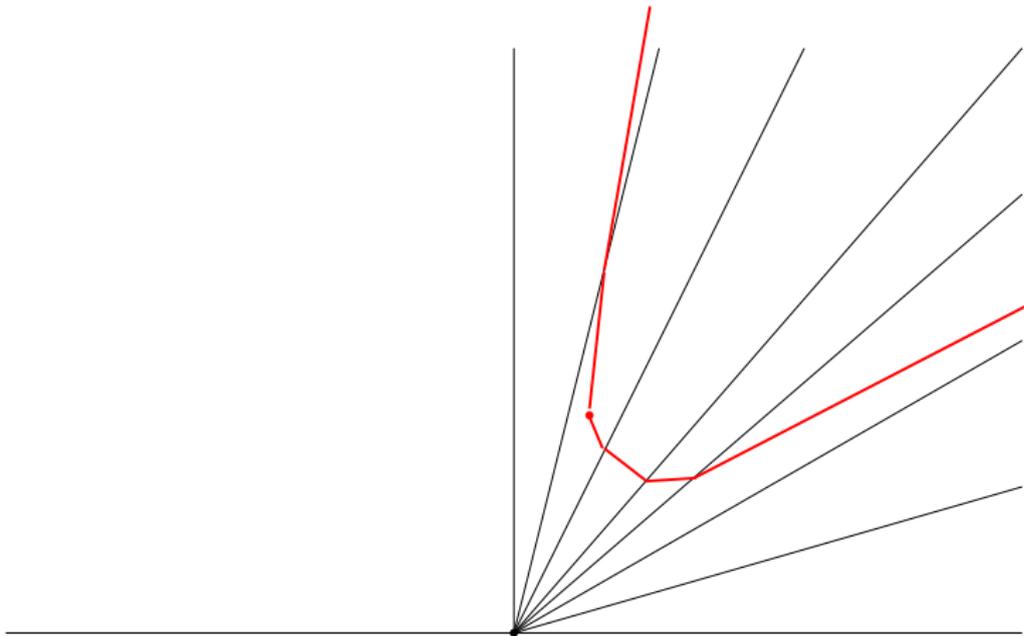
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For every $p_1, p_2, p \in B(\mathbb{Z})$ and $Q \in B$ generic close to p, define

$$C_{p_1,p_2}^{\mathfrak{D},p}(Q) := \sum_{(\gamma_1,\gamma_2)} c(\gamma_1)c(\gamma_2)A^{2\det(s(\gamma_1),s(\gamma_2))} \in R,$$

where the sum is over pairs (γ_1, γ_2) of quantum broken lines for $\mathfrak{D}$ with charges p_1, p_2 and common endpoint Q, such that writing $c(\gamma_1)z^{s(\gamma_1)}$ and $c(\gamma_2)z^{s(\gamma_2)}$ the final monomials, we have $s(\gamma_1) + s(\gamma_2) = p$.

A scattering diagram $\mathfrak{D}$ is *consistent* if for every $p_1, p_2, p \in B(\mathbb{Z})$, $C_{p_1,p_2}^{\mathfrak{D},p}(Q)$ does not depend on the choice of the point Q, and the product on the free R-module

$$\mathcal{A}_{\mathfrak{D}} := \bigoplus_{p \in B(\mathbb{Z})} R\, \vartheta_p$$

defined by

$$\vartheta_{p_1} \vartheta_{p_2} = \sum_{p \in B(\mathbb{Z})} C_{p_1,p_2}^{\mathfrak{D},p}\, \vartheta_p$$

is associative.

Proof

Strategy: construct a consistent scattering diagram $\mathfrak{D}$ and an isomorphism

$$\varphi \colon \mathcal{A}_{\mathfrak{D}} \to \mathsf{Sk}_A(\mathbb{S}_{0,4})$$

such that

$$\varphi(\vartheta_p) = \mathbf{T}(\gamma_p)$$

for every $p \in B(\mathbb{Z})$.

Scattering diagram

- Notations:

$$F(r, s, y, x) := 1 + \frac{rx(1 + x^2)}{(1 - A^{-4}x^2)(1 - A^4x^2)} + \frac{yx^2}{(1 - A^{-4}x^2)(1 - A^4x^2)}$$
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$$R_{1,0} := a_1 a_2 + a_3 a_4, \quad R_{0,1} := a_1 a_3 + a_2 a_4, \quad R_{1,1} := a_1 a_4 + a_2 a_3,$$
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- Define a scattering diagram $\mathfrak{D}$ by

$$\text{if } (m, n) = (1, 0) \mod 2, \, f_{m,n} := F(R_{1,0}, R_{0,1}R_{1,1}, y, z^{-(m,n)}),$$
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Results

Theorem 1 (B, 2020)

The scattering diagram $\mathfrak{D}$ is consistent.

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There exists an isomorphism

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Mirror symmetry

- $\text{Sk}_A(\mathbb{S}_{0,4})$ is a 4-parameters family of non-commutative cubic surfaces, deformation quantization of a 4-parameters family of affine cubic surfaces.

- This 4-parameters family of affine cubic surfaces is the mirror family of the log Calabi-Yau surface (Y, D), where Y is a smooth projective surface in $\mathbb{P}^3$ over $\mathbb{C}$ and D is a triangle of lines on Y. (Gross-Hacking-Keel (2011), Gross-Hacking-Keel-Siebert (2019)). Construction in terms of scattering diagrams constructed in terms of genus 0 log Gromov-Witten invariants of (Y, D).

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- Two deformation quantizations of the 4-family of affine cubic surfaces: the skein algebra and the "quantum mirror" of (Y, D). Natural to compare them.

Cubic surfaces

Y: smooth projective cubic surface in $\mathbb{P}^3$ over $\mathbb{C}$. It is known since Cayley and Salmon (1849) that Y contains 27 lines and that they are 45 triangle configurations of lines. We fix $D = D_1 \cup D_2 \cup D_3$ one of such triangle. The divisor D is anticanonical, and so the pair (Y, D) is log Calabi-Yau.
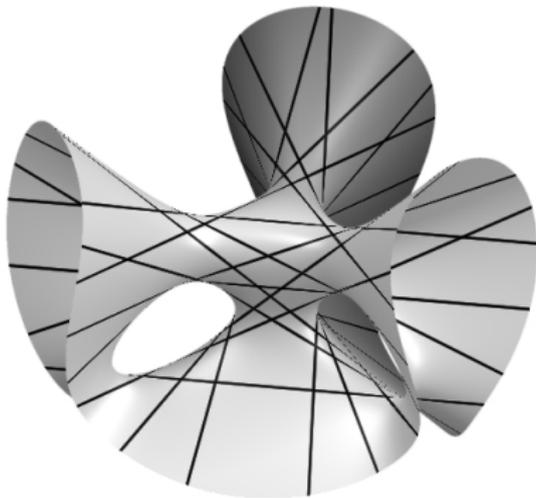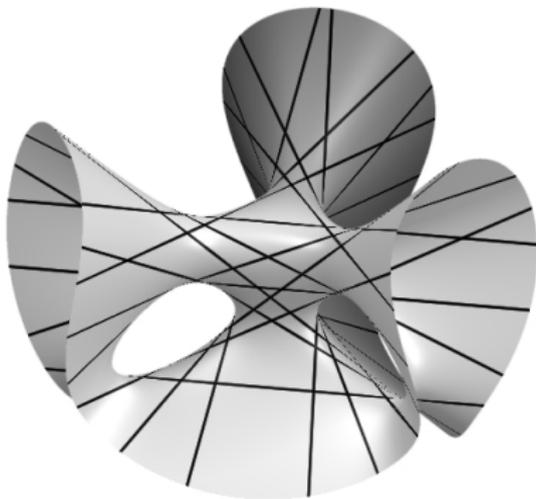
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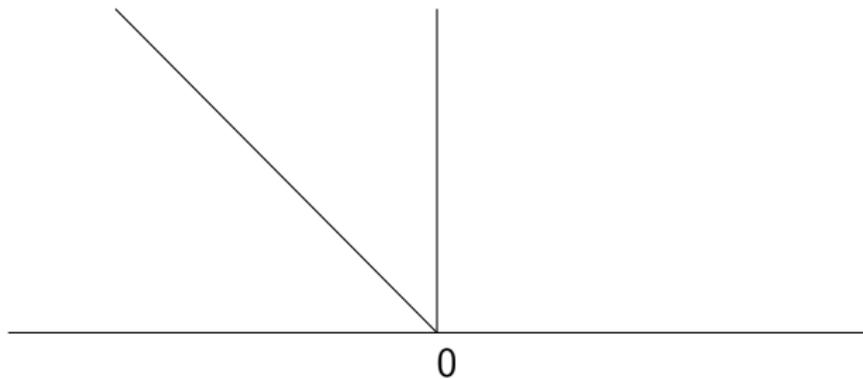
$B := \mathbb{R}^2/\langle \pm id \rangle$ is the dual intersection complex of (Y, D). The set $B(\mathbb{Z})$ of integral points define tangency conditions for curves in (Y, D).

Log Gromov-Witten invariants

- $g \geq 0$, $\beta \in NE(Y)$, $v \in B(\mathbb{Z})$: $N_{g,v}^{\beta} \in \mathbb{Q}$, log Gromov-Witten invariant counting genus g stable maps to (Y, D) of class β with contact order v with D, and insertion of $(-1)^g \lambda_g$. "counts" of genus g holomorphic curves in Y of class β intersecting D in a single point (logarithmic Gromov-Witten invariants).

- Define a scattering diagram $\mathfrak{D}_{(Y,D)}$ on B over $\mathbb{Q}[[\hbar]][NE(Y)]$ by attaching to the ray of direction (m, n) the generating series

$$f_{m,n} := \exp \left(\sum_{k \geq 1} \sum_{\beta \in NE(Y)} \sum_{g \geq 0} 2 \sin \left(\frac{k\hbar}{2} \right) N_{g,k(m,n)}^{\beta} \hbar^{2g-1} t^{\beta} z^{-k(m,n)} \right) .$$

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Log Gromov-Witten invariants

Theorem (B., 2018)

The scattering diagram $\mathfrak{D}_{(Y,D)}$ is consistent.

The genus 0 limit was previously known by Gross-Hacking-Keel (proof relying on the "tropical vertex" paper of Gross-Pandharipande-Siebert).

Theorem (B., 2018)

After explicit specializations and identifications of variables, we have $\mathfrak{D}_{(Y,D)} \simeq \mathfrak{D}$.

Genus 0 case treated by Gross-Hacking-Keel-Siebert (2019).

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- $f_{1,0}$: look at curves intersecting D_1 in one point, and not intersecting D_2 and D_3. The are such $8 = (27 - 3)/3$ lines L_m, intersecting D_1 with contact order 1. There are also two such curves of class $D_2 + D_3$, intersecting D_1 with contact order 2 (strict transforms of the 2 conics passing through 4 given points and tangent to one given line in $\mathbb{P}^2$).

$$f_{(1,0)} =$$

$$\frac{\prod_{m=1}^{8}(1 + t^{L_m}z^{-(1,0)})}{(1 - A^{-4}t^{D_2+D_3}z^{-(2,0)})(1 - t^{D_2+D_3}z^{-(2,0)})^2(1 - A^4 t^{D_2+D_3}z^{-(2,0)})}$$

where $A^4 = q = e^{i\hbar}$.

- The numerator comes from multicovers of the 8 lines L_m and the denominator comes from multicovers of the 2 curves of class $D_2 + D_3$.

- Multicover contributions of lines are "easy" (quiver moduli: point). Multicover contributions of curves of class $D_2 + D_3$ are more sublte (quiver moduli: $\mathbb{P}^1$).

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- $f_{m,n}$ for general $(m, n) \in B(\mathbb{Z})$ primitive?

- $PSL_2(\mathbb{Z})$ of log birational automorphisms of (Y, D): S acts by cyclic permutation of D_1, D_2, D_3, and T acts by blow-up of $D_1 \cap D_2$ and blow-down of D_3.

- Conclude using the invariance of log Gromov-Witten invariants under log birational modifications (Abramovich, Wise, (2013)).

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Thank you for your attention!